

Minimal Assumption Derivation of a weak Clauser-Horne Inequality

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Abstract

According to Bell's theorem a large class of hidden-variable models obeying Bell's notion of local causality conflict with the predictions of quantum mechanics. Recently, a Bell-type theorem has been proven using a weaker notion of local causality, yet assuming the existence of perfectly correlated event types. Here we present a similar Bell-type theorem without this latter assumption. The derived inequality differs from the Clauser-Horne inequality by some small correction terms, which render it less constraining.

Keywords: Bell's theorem; Reichenbach's Principle of Common Cause; Perfect correlations

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1 Introduction

In this article we continue the work of Graßhoff, Portmann, and Wüthrich (2005)¹ and prove a Bell-type theorem from a still weaker set of assumptions. In contrast to Graßhoff et al. (2005), the weakening is reflected in the derived inequality: We get the Clauser-Horne inequality with small correction terms rendering our inequality less constraining.²

There are many different Bell-type theorems with different aims, the weakening of the assumptions being one among many objectives.³ In order to set the theoretical stage, we would like to recall some works⁴ aiming to minimize the strength of the assumptions and set them in context to our own work (see figure 1).

The experimental context of all these derivations is the EPR-Bohm experiment (see section 2). Furthermore, they all assume a locality and a causality condition⁵ for the observable events in terms of “hidden” variables. In its canonical interpretation, quantum mechanics (QM) violates the locality but not the causality condition. In his seminal derivation, Bell (1964) assumed local determinism (LOC and DET) and, additionally, the existence of perfectly correlated event types (PCORR). Then, Clauser, Horne, Shimony, and Holt (1969) derived the CHSH-inequality—again with LOC and DET, but without PCORR. Moreover, Bell (1971) showed two years later, that the same inequality can even be derived if one replaces the assumption of local determinism with a weaker probabilistic notion, which he dubbed “local causality” (LC) (Bell, 1975), and which was later analyzed by Suppes and Zanotti (1976), van Fraassen (1982) and Jarrett (1984) as a conjunction of a locality and a causality condition. As the long philosophical discussion demonstrates, it is already difficult to find an only necessary condition for probabilistic causation. And already Bell (1975) stressed that other definitions of LC are conceivable. Belnap and Szabó (1996) and Hofer-Szabó, Rédei, and Szabó (1999) showed that Reichenbach’s Principle of Common Cause does indeed suggest another form of causality (PCC), which, together with LOC, includes Bell’s notion only as a special case. They also pointed out, that the existing proofs all assume the stronger notion and that it

¹See also Wüthrich (2004).

²A very similar result for the special case of two-valued common causes was derived independently by Hofer-Szabó (2006).

³E.g. GHZ-type theorem’s (Greenberger, Horne, and Zeilinger (1989)) foremost achievement is simplicity.

⁴For more detailed reviews see e.g. Shimony (2005) and Clauser and Shimony (1978).

⁵By choosing this terminology, we do however not intend to exclude the possibility that there can still be non-local causality, even if only the causality condition is violated (see e.g. Butterfield (1992a), Butterfield (1992b), Jones and Clifton (1993), and Maudlin (1994)).

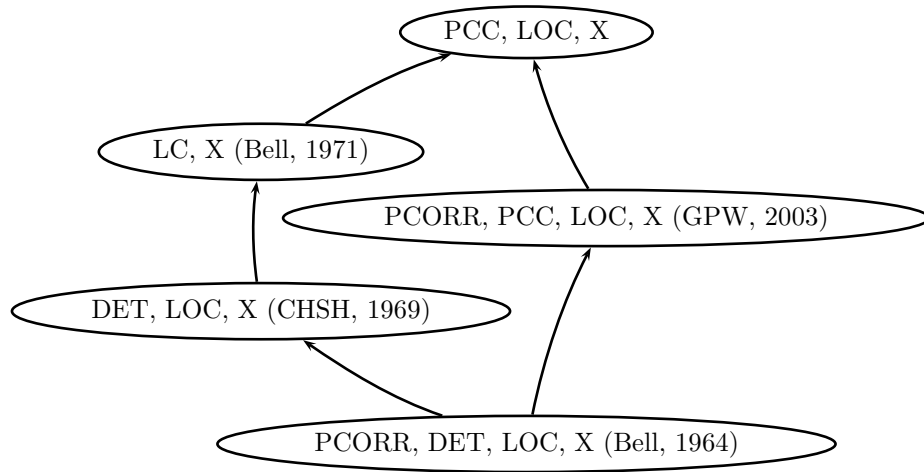


Figure 1: Comparison of the logical strengths of the different sets of assumptions. Each node stands for a set of assumptions from which a Bell-type inequality was derived. If two nodes are connected by an arrow, the set further upwards is a logical implication of the set further downwards but not vice versa. Caption: DET=Determinism, LOC=Locality, PCORR= \exists perfectly correlated event types, LC=local causality, PCC=Reichenbach's principle of common cause, X=further assumptions, shared by all derivations.

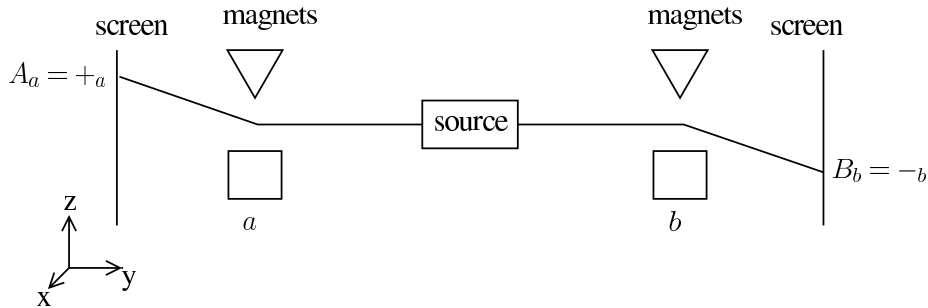


Figure 2: Setup of the EPR-Bohm experiment.

is thus not clear whether a Bell-type theorem can still be proven with PCC. We used PCC as our causality condition in Graßhoff et al. (2005) (GPW) for a proof of a Bell-type theorem, but the minimality of the logical strength of the assumptions was only relative (see figure 1), because we also assumed PCORR. *Given* this assumption, our set of assumptions was minimal. However, there are reasons to think that PCORR is false (see section 3.2), which limits the significance of our result. In this article, we derive a Bell-type inequality without assuming PCORR. Our approach is similarly “straightforward” as the one of Ryff (1997). His intuition “that if a theorem is valid whenever we have perfect correlations, it cannot be totally wrong in the case of almost perfect correlations” can be formulated precisely and proven to be correct in our case.

This article is structured as follows. We describe the EPRB experiment and introduce our notation in section 2. In the main part, section 3, we derive a weak Clauser-Horne inequality. In section 4, we discuss our result and compare it to related work. Specifically, we discuss the significance of the small correction terms in our inequality.

2 The EPRB experiment

Consider the so-called *EPR-Bohm (EPRB) experiment* (Einstein, Podolsky, & Rosen, 1935; Bohm, 1951). Two spin- $\frac{1}{2}$ particles in the *singlet state*

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \quad (1)$$

are separated such that one particle moves to the measurement apparatus of Alice on the left and the other particle to the measurement apparatus of Bob on the right (see figure 2). The experimenter can arbitrarily choose the direction in which the spin is measured with a Stern-Gerlach magnet.

The event⁶ that Alice’s (Bob’s) measurement apparatus is set to measure

⁶If not otherwise stated we refer with “events” always to event *types*, in contrast to event tokens which instantiate types.

the spin in direction a (b) is symbolized by a (b). A_a (B_b) symbolizes the measurement outcome of Alice (Bob) for a measurement in direction a (b). For each direction, there are two possible measurement outcomes: *spin up* ($A_a = +_a$, $B_b = +_b$) and *spin down* ($A_a = -_a$, $B_b = -_b$).⁷ We will always interpret these events also as elements of a Boolean algebra Ω with a classical probability measure p , constituting a classical probability space (Ω, p) . E.g.

$$p(A_a B_b | ab) \quad (2)$$

denotes the probability that Alice' measurement outcome is A_a and Bob's B_b , when measuring in the directions a (Alice) and b (Bob). We will often use the notation⁸

$$p_{a,b}(\dots) := p(\dots | ab), \quad (3)$$

with which we can write (2) as

$$p_{a,b}(A_a B_b). \quad (4)$$

These probabilities are predicted by quantum mechanics as

$$p_{a,b}(+_a +_b) = \frac{1}{2} \sin^2 \frac{\varphi_{a,b}}{2}, \quad (5)$$

$$p_{a,b}(-_a -_b) = \frac{1}{2} \sin^2 \frac{\varphi_{a,b}}{2}, \quad (6)$$

$$p_{a,b}(+_a -_b) = \frac{1}{2} \cos^2 \frac{\varphi_{a,b}}{2}, \quad (7)$$

$$p_{a,b}(-_a +_b) = \frac{1}{2} \cos^2 \frac{\varphi_{a,b}}{2}, \quad (8)$$

where $\varphi_{a,b}$ denotes the angle between the two measurement directions a and b . Also, the outcomes on each side are predicted separately to be completely random:

$$p_{a,b}(A_a) = \frac{1}{2}, \quad (9)$$

$$p_{a,b}(B_b) = \frac{1}{2}. \quad (10)$$

3 Proof of a Bell-type theorem

We will first introduce our assumptions (sections 3.1 and 3.2). In the literature on Bell-type theorems, there is a huge amount of work devoted to the discussion

⁷It is sometimes said that the assumptions of Bell-type theorems are completely independent of QM. However, it seems to us that at least the event structure with A_a , B_b , a and b is adopted from QM. Alternatively, one could try to start from scratch by invoking an independent criterion for event identity. Some steps in this direction have been done by Butterfield (1992b) using David Lewis's account of events.

⁸Note that, if $p(\dots)$ is a probability measure and $p(ab) \neq 0$, then $p_{a,b}(\dots)$ is also a probability measure.

of the assumptions. We do not intend to contribute to this discussion here. Rather, we will discuss the new elements in our set of assumptions. First, we will see that the first three assumptions imply the existence of a common cause which screens off the correlations in question. As in Graßhoff et al. (2005), the new and crucial thing here is that, in contrast to other derivations, we do not demand a single common cause for all the different correlations. Second, since we now have several different common causes, we need to adjust some assumptions (4 and 5) to this new situation. The comments on the remaining assumptions are there for the sake of clarity and not intended to contribute to the ongoing debate.

3.1 Locality and Causality

The correlation between ‘heads up’ and ‘tails down’ when tossing a coin is explained by the identity of the instances of the respective events: Every instance of ‘heads up’ is also an instance of ‘tails down’, and vice versa. Large spatial separation of coinciding instances of A_a and B_b suggests that such is not the case in the EPRB setup:⁹

Assumption 1. *The coinciding instances of the events A_a and B_b are distinct.*

Given this assumption, we can express

Assumption 2. *No A_a or B_b is causally relevant for the other.*

This assumption is supported by the fact that the measurements can be made such that in each run of the experiment the instance of A_a is space-like separated from the instance of B_b . If it were violated and if a cause temporally precedes its effects, the direction of causation would depend on the chosen inertial frame.¹⁰

Assumption 3 (PCC). *If two events A and B with distinct coinciding instances are correlated and neither A is causally relevant for B nor vice versa, then there exists a partition $C = \{C_i\}_{i \in I}$ of Ω , a common cause, such that*

$$p(AB|C_i) = p(A|C_i)p(B|C_i), \quad \forall i \in I.$$

We will assume the cardinality of I to be countable.¹¹ The common cause can alternatively be thought of as a variable (the “hidden” variable) taking on the elements of the partition as values. Thus, when we say “the value of the common cause”, we refer to an element of the partition. In the original formulation, Reichenbach used a partition with two elements, which is here generalized

⁹See also footnote 7.

¹⁰Note however that this is *per se* not a violation of Lorentz invariance and that whether or not this stands in contradiction to the special theory of relativity is an intricate matter. For a discussion, see for example Maudlin (1994) and Weinstein (2006).

¹¹We choose this constraint only for simplicity. The derivation can easily be amended also for I being uncountable.

to a partition with countably many elements.¹²

Now, as can be seen from equations (5)-(10), in general, the event A_a is correlated with event B_b :

$$p_{a,b}(A_a B_b) \neq p_{a,b}(A_a) p_{a,b}(B_b), \text{ except for } \varphi_{a,b} = \frac{\pi}{2} \pmod{\pi}. \quad (11)$$

With assumptions 1 and 2, PCC demands the existence of a common cause $C^{abAB} = \{C_i^{abAB}\}_{i \in I^{abAB}}$ which screens off the correlation:

$$p_{a,b}(A_a B_b | C_i^{abAB}) = p_{a,b}(A_a | C_i^{abAB}) p_{a,b}(B_b | C_i^{abAB}), \quad \forall i \in I^{abAB}. \quad (12)$$

As in Graßhoff et al. (2005) there is a common cause C^{abAB} for *each* quadruple of measurement directions and outcomes (a, b, A_a, B_b) . That is different from other derivations, where a single common cause $\{C_i\}_{i \in I}$ is stipulated for all correlated events:

$$p_{a,b}(A_a B_b | C_i) = p_{a,b}(A_a | C_i) p_{a,b}(B_b | C_i), \quad \forall i \in I. \quad (13)$$

That such a *common* common cause (obeying (13)) was assumed in Bell-type theorems was pointed out and criticized by Belnap and Szabó (1996) and Hofer-Szabó et al. (1999).

Mathematically, (13) is stronger than (12). Indeed, as already pointed out by Butterfield (1989) (p. 123), to get from (12) to (13) one needs at least one further assumption (see also Graßhoff et al. (2005), p. 15 et seq., and Henson (2005), p. 532 et seq.). The additional assumption states that the statistical independence in (12) does not get disrupted if one (additionally to C_i^{abAB}) conditionalizes on other events which are not causally relevant for A_a and B_b . We do not have any strong argument for or against this assumption. To be sure, there are arguments which are brought forward in the literature in favour of it (see e.g. Skyrms (1980), Eells and Sober (1983), and Uffink (1999)).¹³ But we think it is fair to say that the issue is contentious (see e.g. Cartwright (1979)). And, since we only need (12) for our derivation, we do not need to take sides on this question.¹⁴

¹²Reichenbach (1956) and Hofer-Szabó and Rédei (2004) stipulate further conditions, for the two-valued and the general case respectively. For our derivation, we do not need these assumptions, though.

¹³If one aims at a characterization of causality in purely statistical terms, the possibility of disrupting causal independencies by causally irrelevant factors would be a major obstacle (we thank Michael Baumgartner for pointing this out to us). Indeed, one prominent approach in this direction, which is based on Bayesian networks (Spirtes, Glymour, and Scheines (1993), Pearl (2000)) denies this possibility. But of course, its making the sought-after inference from statistics to causality functional is not an argument for its truth.

¹⁴Interestingly, there is a strong argument against this assumption if one embraces the other Reichenbachian conditions (see footnote 12) as well. For a given common cause *system* that obeys the additional assumptions as well, Hofer-Szabó and Rédei (2004) show that there is no finer partitioning possible without disrupting the statistical independence in this finer partition.

Assumption 4 (LOC).

$$p(A_a|abC_i^{abAB}) = p(A_a|aC_i^{abAB}), \quad (14)$$

$$p(B_b|abC_i^{abAB}) = p(B_b|bC_i^{abAB}). \quad (15)$$

This assumptions is meant to prevent the possibility of superluminal causation. Rather than to justify LOC, we will recall the justification of the analogue of it in the traditional derivations and show that the same justification works in our case as well. In this way, we back up our minimality claim.

In the traditional derivations, there is just one common cause. The condition then reads

$$\begin{aligned} p(A_a|abC_i) &= p(A_a|aC_i), \quad \text{and} \\ p(B_b|abC_i) &= p(B_b|bC_i) \quad (\text{for all values of } a, b, A_a \text{ and } B_b). \end{aligned} \quad (16)$$

The canonical justification of (16) runs along the following lines. One first notes that the EPRB experiment can be set up such that the measurement outcome A_a (B_b) and the choice of the measurement setting b (a) are space-like separated. If Alice knew the value of the common cause (which, say, is part of her past light-cone) and (16) did *not* hold, Bob could send Alice a signal superluminally by setting up a measurement direction, since this would alter the corresponding probability of Alice' measurement outcome. Now we do no longer have just one single common cause for all correlations but one for each. Nonetheless, the justification given above works all the same. In the sentence

“If Alice knew *the value of the common cause* (which, say, is part of her past light-cone) and (16) did not hold, Bob could send Alice a signal superluminally by setting up a measurement direction, since this would alter the corresponding probability of Alice' measurement outcome.”,

just replace the italics with “the value of a common cause”.¹⁵

3.2 Common causes for the maximal correlations

In their derivation Graßhoff et al. (2005) exploit that the screening-off condition entails that common causes of perfect correlations determine the effects. The slightest deviation from

$$p_{a=b}(+a|-b) = p_{a=b}(+b|-a) = 1. \quad (17)$$

leads to a breakdown of that type of derivation. Of course, equation (17) is true according to QM and any apparent violation in actual experiments may be attributed to experimental shortcomings, for instance that, in practice, the measurement devices are never set up perfectly parallel.

¹⁵The justification also works if, instead of the value of a single common cause, one takes conjunctions or disjunctions of the values of common causes, or any element of the subalgebra generated by them.

Nevertheless, we would like to do without this assumption. Our motivation for this is twofold. First, there are theoretical grounds on which to expect a violation of the quantum mechanical prediction of perfect correlations. Theoretical work in the different approaches to quantum gravity suggests that tiny violations of Lorentz group invariance are to be expected.¹⁶ Seen as an implication of rotation invariance, (17) would not be warranted any more. The second motivation has to do with the prominent claim that Bell-type theorems rule out the existence of empirically adequate local hidden-variable models on empirical grounds alone. However, if besides the assumptions that define the model as a local hidden-variable model, the only constraint were empirical adequacy, PCORR should not be assumed, because small violations of it are consistent with empirical data.¹⁷

These considerations motivate a weakening of (17) such that we just take the maximal correlations available, without assuming that they are perfect. We do this as follows. For each pair of measurement directions (a, b) , we parametrize the conditional probabilities $p_{a,b}(+a|-b)$ and $p_{a,b}(+b|-a)$ as

$$\begin{aligned} p_{a,b}(+a|-b) &= 1 - \epsilon_{a,b}, \\ p_{a,b}(+b|-a) &= 1 - \epsilon_{b,a}, \text{ with } \epsilon_{a,b}, \epsilon_{b,a} \in [0, 1]. \end{aligned} \quad (18)$$

We will call the set of all measurement directions of Alice (of Bob) D_A (D_B). For each measurement direction $a \in D_A$ ($b \in D_B$), we pick out the measurement direction $\dot{a} \in D_B$ ($\dot{b} \in D_A$) for which $p_{a,b}(+a|-b)$ ($p_{a,b}(+b|-a)$) takes on its maximal value, or, equivalently, $\epsilon_{a,b}$ ($\epsilon_{b,a}$) takes on its minimal value.¹⁸ If the same minimal value is taken on for more than one direction, we make an arbitrary choice. We denote this minimal value with ϵ_a (ϵ_b):¹⁹

$$\begin{aligned} \epsilon_a &:= \min_b \{\epsilon_{a,b}\}, \\ \epsilon_b &:= \min_a \{\epsilon_{b,a}\}. \end{aligned} \quad (19)$$

Thus, we have

$$\begin{aligned} p_{a,\dot{a}}(+a|-a) &= 1 - \epsilon_a, \\ p_{\dot{b},b}(+b|-b) &= 1 - \epsilon_b. \end{aligned} \quad (20)$$

Because of assumptions 1 to 3, we have (in the notation of formula (12)) a common cause $\{C_i^{a\dot{a}+-}\}_{i \in I^{a\dot{a}+-}}$ ($\{C_i^{b\dot{b}-+}\}_{i \in I^{b\dot{b}-+}}$) for the events $+a$ and $-\dot{a}$ ($+b$ and $-\dot{b}$). Henceforth, we will use the short hand $\{C_i^a\}_{i \in I^a}$ ($\{C_i^b\}_{i \in I^b}$) for

¹⁶See e.g. Mattingly (2005) for references.

¹⁷In the context of the Kochen-Specker theorem, a similar loophole was exploited to construct a non-contextual empirical adequate model by Clifton and Kent (2000).

¹⁸If the number of measurement directions (i.e. the cardinality of D_A and D_B) is not finite, it is possible, that there is no such minimal value but only an infimum. The proof can be amended also for this case, but we will refrain from doing this here.

¹⁹Note that we do not assume that the minimal value is taken on for parallel measurement directions.

$\{C_i^{a\dot{a}+-}\}_{i \in I^{a\dot{a}+-}}$ ($\{C_i^{bb--}\}_{i \in I^{bb--}}$). With this notation, we get:

$$\begin{aligned} p_{a,\dot{a}}(+_a, -_a | C_i^a) &= p_{a,\dot{a}}(+_a | C_i^a) p_{a,\dot{a}}(-_a | C_i^a), \forall i \in I^a, \\ p_{\dot{b},b}(-_b, +_b | C_i^b) &= p_{\dot{b},b}(-_b | C_i^b) p_{\dot{b},b}(+_b | C_i^b), \forall i \in I^b. \end{aligned} \quad (21)$$

Assumption 5.

$$p(aC_i^a) = p(a)p(C_i^a) \quad (22)$$

$$p(bC_i^b) = p(b)p(C_i^b) \quad (23)$$

$$p(abC_i^a) = p(ab)p(C_i^a), \quad (24)$$

$$p(abC_i^b) = p(ab)p(C_i^b), \quad (25)$$

$$p(abC_i^a C_j^b) = p(ab)p(C_i^a C_j^b). \quad (26)$$

With this assumption one would like to exclude that the common causes are causally relevant for the setting of the measurement apparatuses or vice versa. Furthermore, one would like to exclude a common cause for these factors.

3.3 Constraints for $p_{a,b}(+_a+_b)$, $p(+_a|a)$, and $p(+_b|b)$

To obtain a Bell-type inequality we need an upper and a lower bound for

$$p_{a,b}(+_a+_b), p(+_a|a), \text{ and } p(+_b|b). \quad (27)$$

We will need the following proposition.

Proposition 1. *Let two events A and B with $p(A) = p(B) = 0.5$ be almost perfectly correlated ($p(A|B) = 1 - \epsilon$) and assume a common cause $C = \{C_i\}_{i \in I}$, such that*

$$p(AB|C_i) = p(A|C_i)p(B|C_i), \forall i \in I. \quad (28)$$

Then

$$\sum_{i \in I_1} p(C_i) - \sqrt{\epsilon} \leq p(A) < \sum_{i \in I_1} p(C_i) + 4\sqrt{\epsilon} - 2\epsilon, \quad (29)$$

where

$$I_1 := \{i \in I : p(A|C_i) \geq 1 - \sqrt{\epsilon}\}. \quad (30)$$

We prove proposition 1 in appendix A.

With the definition

$$C := \vee_{i \in I_1} C_i, \quad (31)$$

equation (29) reads

$$p(C) - \sqrt{\epsilon} \leq p(A) < p(C) + 4\sqrt{\epsilon} - 2\epsilon, \quad (32)$$

or, equivalently,

$$p(A) - 4\sqrt{\epsilon} + 2\epsilon < p(C) \leq p(A) + \sqrt{\epsilon}. \quad (33)$$

We define

$$\begin{aligned} I_1^a &:= \{i \in I^a : p_{a,\dot{a}}(+_a|C_i^a) \stackrel{(*)}{=} p(+_a|aC_i^a) \geq 1 - \sqrt{\epsilon_a}\}, \\ C^a &:= \vee_{i \in I_1^a} C_i^a, \\ I_1^b &:= \{i \in I^b : p_{b,\dot{b}}(+_b|bC_i^b) \stackrel{(*)}{=} p(+_b|bC_i^b) \geq 1 - \sqrt{\epsilon_b}\}, \\ C^b &:= \vee_{i \in I_1^b} C_i^b. \end{aligned} \quad (34)$$

In (*), we use LOC.

With the substitutions

$$\begin{aligned} p(\dots) &\rightarrow p_{a,\dot{a}}(\dots), & p(\dots) &\rightarrow p_{b,\dot{b}}(\dots), \\ A &\rightarrow +_a, & A &\rightarrow +_b, \\ B &\rightarrow -_{\dot{a}}, & B &\rightarrow -_{\dot{b}}, \\ C &\rightarrow C^a, & C &\rightarrow C^b, \\ \epsilon &\rightarrow \epsilon_a, & \epsilon &\rightarrow \epsilon_b, \end{aligned} \quad (35)$$

we get

$$\begin{aligned} p_{a,\dot{a}}(+_a) - 4\sqrt{\epsilon_a} + 2\epsilon_a &< p_{a,\dot{a}}(C^a) \leq p_{a,\dot{a}}(+_a) + \sqrt{\epsilon_a}, \\ p_{b,\dot{b}}(+_b) - 4\sqrt{\epsilon_b} + 2\epsilon_b &< p_{b,\dot{b}}(C^b) \leq p_{b,\dot{b}}(+_b) + \sqrt{\epsilon_b}. \end{aligned} \quad (36)$$

Using assumption 4, 5, and the definition

$$\epsilon := \max_{a,b} \{\epsilon_a, \epsilon_b\}, \quad (37)$$

we get

$$\begin{aligned} p(+_a|a) - \Delta^+ &< p(C^a) \leq p(+_a|a) + \Delta^-, \\ p(+_b|b) - \Delta^+ &< p(C^b) \leq p(+_b|b) + \Delta^-, \end{aligned} \quad (38)$$

with

$$\begin{aligned} \Delta^+ &= 4\sqrt{\epsilon} - 2\epsilon, \\ \Delta^- &= \sqrt{\epsilon}. \end{aligned} \quad (39)$$

Using again assumptions 4 and 5, the following bounds for $p(+_a +_b | ab)$ can be derived (this is shown in appendix B):

$$p_{a,b}(+_a +_b) - \Delta_{a,b}^+ < p(C^a C^b) \leq p_{a,b}(+_a +_b) + \Delta_{a,b}^-, \quad (40)$$

with

$$\begin{aligned} \Delta_{a,b}^+ &= \frac{(p(a) + p(b))(5\sqrt{\epsilon} - 2\epsilon)}{p(ab)}, \\ \Delta_{a,b}^- &= \frac{(p(a) + p(b))\sqrt{\epsilon}}{p(ab)}. \end{aligned} \quad (41)$$

3.4 A weak Clauser-Horne inequality

In the next step, we make use of a constraint, which holds for the probabilities of arbitrary events. For events A and B to be elements of a classical probability space, it is not enough that

$$\begin{aligned} 0 &\leq p(A) \leq 1, \\ 0 &\leq p(B) \leq 1, \\ 0 &\leq p(AB) \leq 1, \end{aligned} \tag{42}$$

and

$$\begin{aligned} p(AB) &\leq p(A), \\ p(AB) &\leq p(B). \end{aligned} \tag{43}$$

We note first, that (“ \bar{A} ” means “not A ”)

$$p(AB) + p(A\bar{B}) + p(\bar{A}B) + p(\bar{A}\bar{B}) = 1. \tag{44}$$

It is also

$$\begin{aligned} p(A) &= p(AB) + p(A\bar{B}), \quad \text{and} \\ p(B) &= p(AB) + p(\bar{A}B), \end{aligned} \tag{45}$$

and hence

$$p(A) + p(B) - p(AB) = p(A\bar{B}) + p(\bar{A}B), \tag{46}$$

which implies with equation (44)²⁰

$$0 \leq p(A) + p(B) - p(AB) \leq 1. \tag{47}$$

For more than two events there are more constraints in the form of such inequalities.²¹ For four events A, A', B and B' , one constraint reads

$$-1 \leq p(AB) + p(AB') + p(A'B') - p(A'B) - p(A) - p(B') \leq 0. \tag{48}$$

This is the Clauser-Horne inequality²² (Clauser & Horne, 1974), which we prove in appendix C. This inequality is an a priori constraint for arbitrary events. Hence, for the measurement directions $1, 2 \in D_A$ and $3, 4 \in D_B$, it is also

$$\begin{aligned} -1 &\leq p(C^1C^3) + p(C^1C^4) + p(C^2C^4) \\ &\quad - p(C^2C^3) - p(C^1) - p(C^4) \\ &\leq 0. \end{aligned} \tag{49}$$

²⁰This can also be seen by noting that $p(A) + p(B) - p(AB)$ is the probability of the disjunction of A and B , $p(A \vee B)$.

²¹For a detailed discussion and the beautiful connection to the geometry of convex polytopes, see e.g. Pitowsky (1989).

²²What Clauser and Horne (1974) have actually derived is inequality (50) without the correction terms (the Δ s). In (50), there are conditional probabilities involved. Nevertheless, we adopt common terminology and refer to both inequalities with the same name, since it will always be clear from the context which is meant.

Together with inequality (40)

$$p_{a,b}(+a+b) - \Delta_{a,b}^+ < p(C^a C^b) \leq p_{a,b}(+a+b) + \Delta_{a,b}^-$$

and inequality (38)

$$\begin{aligned} p(+a|a) - \Delta^+ &< p(C^a) \leq p(+a|a) + \Delta^-, \\ p(+b|b) - \Delta^+ &< p(C^b) \leq p(+b|b) + \Delta^- \end{aligned}$$

one gets

$$\begin{aligned} -1 - \Delta_{1,3}^- - \Delta_{1,4}^- - \Delta_{2,4}^- - \Delta_{2,3}^+ - 2\Delta^+ \\ < p_{1,3}(+1+3) + p_{1,4}(+1+4) + p_{2,4}(+2+4) \\ &\quad - p_{2,3}(+2+3) - p(+1|1) - p(+4|4) \\ < \Delta_{1,3}^+ + \Delta_{1,4}^+ + \Delta_{2,4}^+ + \Delta_{2,3}^- + 2\Delta^-, \end{aligned} \quad (50)$$

with

$$\begin{aligned} \Delta_{a,b}^- &= \frac{(p(a) + p(b))\sqrt{\epsilon}}{p(ab)}, \\ \Delta_{a,b}^+ &= \frac{(p(a) + p(b))(5\sqrt{\epsilon} - 2\epsilon)}{p(ab)}, \\ \Delta^- &= \sqrt{\epsilon}, \\ \Delta^+ &= 4\sqrt{\epsilon} - 2\epsilon. \end{aligned} \quad (51)$$

Note that this inequality reduces to the Clauser-Horne inequality for $\epsilon = 0$.

3.5 Contradiction

The predicted values of $p_{1,3}(+1, +3)$, $p_{1,4}(+1, +4)$, $p_{2,4}(+2, +4)$ and $p_{2,3}(+2, +3)$ by QM are such that the maximal violation²³ for the lower bound of (50) occurs (among others) for the angles $\varphi_{1,3} = \varphi_{1,4} = \varphi_{2,4} = \frac{\pi}{4}$ and $\varphi_{2,3} = \frac{3\pi}{4}$:

$$\begin{aligned} -1 - \Delta_{1,3}^- - \Delta_{1,4}^- - \Delta_{2,4}^- - \Delta_{2,3}^+ - \Delta^+ - \Delta^+ \\ < p_{1,3}(+1+3) + p_{1,4}(+1+4) + p_{2,4}(+2, +4) \\ &\quad - p_{2,3}(+2+3) - p(+1|1) - p(+4|4) = -\frac{\sqrt{2}+1}{2}. \end{aligned} \quad (52)$$

The maximal violation for the upper bound occurs (among others) for the angles $\varphi_{1,3} = \varphi_{2,4} = \frac{3\pi}{4}$, $\varphi_{1,4} = \frac{5\pi}{4}$ and $\varphi_{2,3} = \frac{\pi}{4}$:

$$\begin{aligned} \frac{\sqrt{2}-1}{2} &= p_{1,3}(+1+3) + p_{1,4}(+1+4) + p_{2,4}(+2+4) \\ &\quad - p_{2,3}(+2+3) - p(+1|1) - p(+4|4) \\ &< \Delta_{1,3}^+ + \Delta_{1,4}^+ + \Delta_{2,4}^+ + \Delta_{2,3}^- + \Delta^- + \Delta^-, \end{aligned} \quad (53)$$

²³These violations are also maximal in that no other quantum mechanical two-particle state for two spin- $\frac{1}{2}$ -particles yields a bigger violation (see Tsirelson (Cirel'son) (1980) and Cabello (2002)).

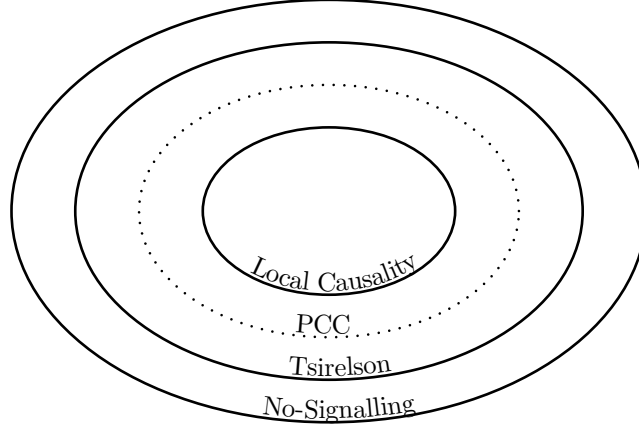


Figure 3: Comparison with other constraints.

With $p(ab) = \frac{1}{4}$ and $p(a) = p(b) = \frac{1}{2}$, one has

$$\Delta_{a,b}^- = 4\sqrt{\epsilon} \quad \text{and} \quad \Delta_{a,b}^+ = 20\sqrt{\epsilon} - 8\epsilon. \quad (54)$$

With the chosen angles and measurement probabilities one gets

$$\frac{\sqrt{2}-1}{2} < 40\sqrt{\epsilon} - 12\epsilon. \quad (55)$$

for the lower bound. This inequality is violated for

$$\epsilon \leq \epsilon_{\max}^l, \quad \epsilon_{\max}^l = 2.689 \cdot 10^{-5}. \quad (56)$$

The inequality for the upper bound reads

$$\frac{\sqrt{2}-1}{2} < 66\sqrt{\epsilon} - 24\epsilon, \quad (57)$$

which is violated for

$$\epsilon \leq \epsilon_{\max}^u, \quad \epsilon_{\max}^u = 9.869 \cdot 10^{-6}. \quad (58)$$

Thus the quantum mechanical predictions contradict the predictions of a hidden variable model obeying our assumptions for

$$\epsilon \leq \epsilon_{\max}^l = 2.698 \cdot 10^{-5}. \quad (59)$$

4 Discussion

Even though the four sets of assumptions in Bell (1964), Clauser et al. (1969), Bell (1971) and Graßhoff et al. (2005) (see figure 1) differ, they all imply the

same constraints on the correlations as expressed in the Clauser-Horne inequality.²⁴

One of the questions left open by Graßhoff et al. (2005) is what constraints are implied *without* assuming the existence of perfectly correlated events. Since with a slightest deviation from perfect correlations the proof by Graßhoff et al. (2005) breaks down, it gives no hint as to whether the same Bell-type constraints follow nor whether a contradiction to QM is entailed at all. In the present paper we have given a partial answer to that question.

The inequality we get at the end of our derivation is stronger than the quantum mechanical predictions (for $\epsilon \leq \epsilon_{\max}^l$), but weaker than the Clauser-Horne inequality (for $\epsilon > 0$). Thus, the weakening of the assumptions is also reflected in a resulting weakening of the constraints. Note however that we did not prove that this weakening is really an implication of our assumptions. What we have shown is only that the conditional probabilities *at least* have to obey the constraint (50).²⁵ To prove the stronger proposition, one could try to construct a separate common cause model obeying all our assumptions that violates the Clauser-Horne inequality without correction terms, but does not violate the weak Clauser-Horne inequality.

Now, we would like to compare our inequality to other prominent constraints (see figure 3).²⁶ Even though Bell's theorem excludes models which obey local causality, the predictions of QM for $p(A_a B_b | ab)$ still obey the no-signalling constraint²⁷

$$\begin{aligned} \sum_{B_b} p(A_a B_b | ab) &= \sum_{B_b} p(A_a B_b | ab'), \\ \sum_{A_a} p(A_a B_b | ab) &= \sum_{A_{a'}} p(A_{a'} B_b | a'b), \end{aligned} \quad (60)$$

which states that the probability of the measurement outcome on one side does not depend on the measurement direction on the other side (given the quantum mechanical state of the system). Moreover, there are some correlations obeying no-signalling which are not permitted by quantum mechanics.²⁸ The bounds, which are allowed by quantum mechanics, were first derived by Tsirelson (Cirel'son) (1980). Notoriously, still more constraining are the Bell inequalities. The situation is drawn schematically in figure 3. The boarder at the margin is the least constraining coming from the no-signalling condition (60). Next is the Tsirelson-bound, which is again weaker than the bound coming

²⁴Even though the derived inequalities have a different form in Bell (1964), Bell (1971) and Graßhoff et al. (2005), the assumptions are all sufficient to derive also the Clauser-Horne inequality.

²⁵This proviso is also necessary, because some steps can be optimized in our derivation. For example, one can choose the borders of the partitions in (64) and (79) differently, such that one would get tighter constraints, that is, the correction terms to the Clauser-Horne inequality (the Δ 's) would become smaller.

²⁶For an overview, see e.g. Gisin (2005).

²⁷See e.g. Redhead (1987), pp. 113-117 and references therein.

²⁸By this we mean possible predictions for the probabilities of $p_{a,b}(A_a B_b)$ coming from Hilbert space-vectors.

from the Clauser-Horne inequality (local causality). The bound coming from inequality (50) lies between the Tsirelson bound and the bound coming from the Clauser-Horne inequality, depending on the value of ϵ . For $\epsilon \neq 0$ there are quantum mechanical states which do violate the Clauser-Horne inequality but not (50). This reveals the following a priori possibility. As Gisin (1991) showed, the correlations coming from pure entangled states always violate the Clauser-Horne inequality. For $\epsilon \neq 0$, Gisin's argument is not sufficient to conclude that all entangled states violate inequality (50). Hence, it is an open question, whether or not there exist models obeying all our assumptions, for the correlations of some entangled pure states.

Even though ϵ_{\max}^l is not zero, it is still very small. Particularly, violations of correlations deviating only through ϵ_{\max}^l from being perfect, can experimentally not be ruled out. This means that we cannot rule out the existence of an empirically adequate hidden variable model obeying all our assumptions. On the other hand, from a theoretical point of view, a deviation from perfect correlation of order ϵ_{\max}^l is rather big. Modulo some theoretical assumptions, any non-vanishing ϵ can be interpreted as a violation of rotation invariance (see section 3.3), which moreover induces a violation of Lorentz invariance.²⁹ Triggered by theoretical works in various approaches to quantum gravity, which either imply violations of Lorentz invariance or render such a violation natural, there has been a tremendous experimental effort for finding signatures of such violations during the last ten years or so (for a recent review, see e.g. Mattingly (2005)). The constraints coming from negative results of such experiments are rather strong. In view of these findings, one would expect ϵ to be smaller than ϵ_{\max}^l and the inequality (50) to be violated.

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A Proof of proposition 1

Proposition 1. *Let two events A and B with $p(A) = p(B) = 0.5$ be almost perfectly correlated ($p(A|B) = 1 - \epsilon$) and assume a common cause $C = \{C_i\}_{i \in I}$, such that*

$$p(AB|C_i) = p(A|C_i)p(B|C_i), \quad \forall i \in I. \quad (61)$$

²⁹Whether or not a violation under rotation invariance implies also a violation of Lorentz boost invariance is model dependent (see Mattingly (2005)).

Then

$$\sum_{i \in I_1} p(C_i) - \sqrt{\epsilon} \leq p(A) < \sum_{i \in I_1} p(C_i) + 4\sqrt{\epsilon} - 2\epsilon, \quad (62)$$

where

$$I_1 := \{i \in I : p(A|C_i) \geq 1 - \sqrt{\epsilon}\}. \quad (63)$$

Proof. We will partition I into the following three subsets:

$$\begin{aligned} I_1 &:= \{i \in I : p(A|C_i) \geq 1 - \sqrt{\epsilon}\}, \\ I_2 &:= \{i \in I : \sqrt{\epsilon} < p(A|C_i) < 1 - \sqrt{\epsilon}\}, \\ I_3 &:= \{i \in I : p(A|C_i) \leq \sqrt{\epsilon}\}. \end{aligned} \quad (64)$$

It is

$$p(A) = \sum_{i \in I_1} p(A|C_i)p(C_i) + \sum_{i \in I_2} p(A|C_i)p(C_i) + \sum_{i \in I_3} p(A|C_i)p(C_i). \quad (65)$$

With the definitions (64) the following inequalities hold:

$$\begin{aligned} p(A) &\geq \sum_{i \in I_1} p(A|C_i)p(C_i) \geq (1 - \sqrt{\epsilon}) \sum_{i \in I_1} p(C_i) \geq \sum_{i \in I_1} p(C_i) - \sqrt{\epsilon}, \\ p(A) &\leq \sum_{i \in I_1} p(C_i) + \sum_{i \in I_2} p(A|C_i)p(C_i) + \sqrt{\epsilon}. \end{aligned} \quad (66)$$

Hence, to complete the proof we have to show that

$$\sum_{i \in I_2} p(A|C_i)p(C_i) < 3\sqrt{\epsilon} - 2\epsilon. \quad (67)$$

It is

$$\begin{aligned} \frac{\epsilon}{2} &= \underbrace{p(A)}_{\frac{1}{2}} - \underbrace{p(AB)}_{\frac{1-\epsilon}{2}} = \sum_{i \in I} [p(A|C_i) - p(AB|C_i)] p(C_i) \\ &\stackrel{(*)}{=} \sum_{i \in I} [p(A|C_i) - p(A|C_i)p(B|C_i)] p(C_i) \\ &= \sum_{i \in I} p(A|C_i) [1 - p(B|C_i)] p(C_i), \end{aligned} \quad (68)$$

where we used (61) to get equality (*). Since everything is symmetric in A and B , the same holds if one exchanges A and B for each other. We thus have

$$\sum_{i \in I} p(A|C_i) [1 - p(B|C_i)] p(C_i) = \frac{\epsilon}{2}, \quad (69)$$

$$\sum_{i \in I} p(B|C_i) [1 - p(A|C_i)] p(C_i) = \frac{\epsilon}{2}. \quad (70)$$

Since all terms in the sums on the L.H.S. of eq. (69, 70) are positive, the following inequalities hold for all subsets I^\subset of the value space I :

$$0 \leq \sum_{i \in I^\subset} p(A|C_i) [1 - p(B|C_i)] p(C_i) \leq \frac{\epsilon}{2}, \quad \forall I^\subset \subset I, \quad (71)$$

$$0 \leq \sum_{i \in I^\subset} p(B|C_i) [1 - p(A|C_i)] p(C_i) \leq \frac{\epsilon}{2}, \quad \forall I^\subset \subset I. \quad (72)$$

Subtracting (71) from (72), one gets

$$\left| \sum_{i \in I^\subset} [p(A|C_i) - p(B|C_i)] p(C_i) \right| \leq \frac{\epsilon}{2}, \quad \forall I^\subset \subset I. \quad (73)$$

With the definitions

$$I_2^{A \geq B} := \{i \in I_2 : p(A|C_i) \geq p(B|C_i)\}, \quad (74)$$

$$I_2^{A < B} := \{i \in I_2 : p(A|C_i) < p(B|C_i)\} \quad (75)$$

and applying (73) for these sets, one gets

$$\left| \sum_{i \in I_2^{A \geq B}} [p(A|C_i) - p(B|C_i)] p(C_i) \right| = \sum_{i \in I_2^{A \geq B}} |p(A|C_i) - p(B|C_i)| p(C_i) \leq \frac{\epsilon}{2}, \quad (76)$$

$$\left| \sum_{i \in I_2^{A < B}} [p(A|C_i) - p(B|C_i)] p(C_i) \right| = \sum_{i \in I_2^{A < B}} |p(A|C_i) - p(B|C_i)| p(C_i) \leq \frac{\epsilon}{2}. \quad (77)$$

Adding these two inequalities, one gets:

$$\sum_{i \in I_2} |p(A|C_i) - p(B|C_i)| p(C_i) \leq \epsilon. \quad (78)$$

We partition I_2 in the following two subsets:

$$\begin{aligned} I_2^{\geq \sqrt{\epsilon}} &:= \left\{ i \in I_2 : |p(A|C_i) - p(B|C_i)| \geq \frac{\sqrt{\epsilon}}{2} \right\}, \\ I_2^{< \sqrt{\epsilon}} &:= \left\{ i \in I_2 : |p(A|C_i) - p(B|C_i)| < \frac{\sqrt{\epsilon}}{2} \right\}. \end{aligned} \quad (79)$$

From

$$\sum_{i \in I_2^{\geq \sqrt{\epsilon}}} |p(A|C_i) - p(B|C_i)| p(C_i) \geq \frac{\sqrt{\epsilon}}{2} \sum_{i \in I_2^{\geq \sqrt{\epsilon}}} p(C_i) \quad (80)$$

together with (78), we get

$$\sum_{i \in I_2^{\geq \sqrt{\epsilon}}} p(C_i) \leq 2\sqrt{\epsilon}. \quad (81)$$

Remember, that we want to derive an upper bound for

$$\sum_{i \in I_2} p(A|C_i)p(C_i). \quad (82)$$

With (81), we already have

$$\begin{aligned} \sum_{i \in I_2} p(A|C_i)p(C_i) &= \sum_{i \in I_2^{\geq \sqrt{\epsilon}}} p(A|C_i)p(C_i) + \sum_{i \in I_2^{< \sqrt{\epsilon}}} p(A|C_i)p(C_i) \\ &< (1 - \sqrt{\epsilon})2\sqrt{\epsilon} + \sum_{i \in I_2^{< \sqrt{\epsilon}}} p(A|C_i)p(C_i). \end{aligned} \quad (83)$$

We will use again inequality (71), this time for the set $I_2^{< \sqrt{\epsilon}}$:

$$\sum_{i \in I_2^{< \sqrt{\epsilon}}} p(A|C_i) [1 - p(B|C_i)] p(C_i) \leq \frac{\epsilon}{2}. \quad (84)$$

Because we are looking at the subset $I_2^{< \sqrt{\epsilon}}$, it is

$$\sum_{i \in I_2^{< \sqrt{\epsilon}}} p(A|C_i) [1 - p(B|C_i)] p(C_i) > \sum_{i \in I_2^{< \sqrt{\epsilon}}} p(A|C_i) \left[1 - p(A|C_i) - \frac{\sqrt{\epsilon}}{2} \right] p(C_i). \quad (85)$$

With (84), one gets

$$\sum_{i \in I_2^{< \sqrt{\epsilon}}} p(A|C_i) \left[1 - p(A|C_i) - \frac{\sqrt{\epsilon}}{2} \right] p(C_i) < \frac{\epsilon}{2}. \quad (86)$$

Now, since $I_2^{< \sqrt{\epsilon}}$ is a subset of I_2 , $p(A|C_i)$ takes on values in the interval $[\sqrt{\epsilon}, 1 - \sqrt{\epsilon}]$. One can check that each summand is certainly greater than for $p(A|C_i) = 1 - \sqrt{\epsilon}$. We have

$$\sum_{i \in I_2^{< \sqrt{\epsilon}}} p(A|C_i) \left[1 - p(A|C_i) - \frac{\sqrt{\epsilon}}{2} \right] p(C_i) > (1 - \sqrt{\epsilon}) \frac{\sqrt{\epsilon}}{2} \sum_{i \in I_2^{< \sqrt{\epsilon}}} p(C_i). \quad (87)$$

We get the constraint

$$\sum_{i \in I_2^{< \sqrt{\epsilon}}} p(C_i) < \frac{\sqrt{\epsilon}}{(1 - \sqrt{\epsilon})}. \quad (88)$$

With (83) one gets

$$\sum_{i \in I_2} p(A|C_i)p(C_i) < (1 - \sqrt{\epsilon})2\sqrt{\epsilon} + \sqrt{\epsilon} = 3\sqrt{\epsilon} - 2\epsilon, \quad (89)$$

which is what we wanted to show. We have

$$\sum_{i \in I_1} p(C_i) - \sqrt{\epsilon} \leq p(A) < \sum_{i \in I_1} p(C_i) + 4\sqrt{\epsilon} - 2\epsilon. \quad (90)$$

□

B Bounds for $p(+_a +_b | ab)$

With (38) and assumption 5, we get

$$\begin{aligned} p(+_a | a) &= p(+_a C^a | a) + p(+_a \overline{C^a} | a) < p(C^a | a) + 4\sqrt{\epsilon} - 2\epsilon \\ &= p(+_a C^a | a) + p(\overline{+}_a C^a | a) + 4\sqrt{\epsilon} - 2\epsilon, \end{aligned} \quad (91)$$

and hence

$$p(+_a \overline{C^a}) < p(\overline{+}_a C^a) + p(a) (4\sqrt{\epsilon} - 2\epsilon). \quad (92)$$

Furthermore, we have

$$\begin{aligned} p(\overline{+}_a C^a | a) &\stackrel{(*)}{=} \sum_{i \in I_1^a} p(\overline{+}_a | a C_i^a) p(C_i^a) \\ &= \sum_{i \in I_1^a} (1 - p(+_a | a C_i^a)) p(C_i^a) \\ &\leq \sqrt{\epsilon} \sum_{i \in I_1^a} p(C_i^a) \leq \sqrt{\epsilon}, \end{aligned} \quad (93)$$

where we used assumption 5 to get equality (*). We can write (93) as

$$p(\overline{+}_a C^a) \leq p(a) \sqrt{\epsilon}, \quad (94)$$

such that we get from (92)

$$p(+_a \overline{C^a} X) < p(a) (5\sqrt{\epsilon} - 2\epsilon), \quad (95)$$

because for any X and any Y , $p(XY) \leq p(Y)$. Next, from

$$p(+_a C^a X) = p(a C^a X) - p(\overline{+}_a C^a X) \quad (96)$$

together with (94) and because $p(\overline{+}_a a C^a X) \leq p(\overline{+}_a a C^a)$ we get

$$p(+_a a C^a X) \geq p(a C^a X) - p(a) \sqrt{\epsilon}. \quad (97)$$

Starting from the other inequalities in (38), we get inequalities analogue to (95) and (97). We have

$$p(+_a a \overline{C^a} X) < p(a) (5\sqrt{\epsilon} - 2\epsilon), \quad (98)$$

$$p(+_b b \overline{C^b} X) < p(b) (5\sqrt{\epsilon} - 2\epsilon), \quad (99)$$

$$p(+_a a C^a X) \geq p(a C^a X) - p(a) \sqrt{\epsilon}, \quad (100)$$

$$p(+_b b C^b X) \geq p(b C^b X) - p(b) \sqrt{\epsilon}. \quad (101)$$

Now, we can derive an upper bound for $p(+_a +_b ab)$, using (98) and (99):

$$\begin{aligned} p(+_a +_b ab) &= p(+_a +_b ab C^a) + p(+_a +_b ab \overline{C^a}) \\ &< p(+_a +_b ab C^a) + p(a) (5\sqrt{\epsilon} - 2\epsilon) \\ &< p(+_a +_b ab C^a C^b) + (p(a) + p(b)) (5\sqrt{\epsilon} - 2\epsilon) \\ &\leq p(ab C^a C^b) + (p(a) + p(b)) (5\sqrt{\epsilon} - 2\epsilon). \end{aligned} \quad (102)$$

Using also assumption 5, we finally obtain

$$\begin{aligned} p(+_a +_b | ab) &\equiv \frac{p(+_a +_b ab)}{p(ab)} \\ &< \frac{p(ab C^a C^b) + (p(a) + p(b)) (5\sqrt{\epsilon} - 2\epsilon)}{p(ab)} \\ &= p(C^a C^b | ab) + (p(a) + p(b)) \frac{5\sqrt{\epsilon} - 2\epsilon}{p(ab)} \\ &= p(C^a C^b) + (p(a) + p(b)) \frac{5\sqrt{\epsilon} - 2\epsilon}{p(ab)}. \end{aligned} \quad (103)$$

Next, we derive a lower bound.

$$\begin{aligned} p(+_a +_b ab) &\geq p(+_a +_b ab C^a C^b) \\ &\geq p(+_b ab C^a C^b) - p(a) \sqrt{\epsilon} \\ &\geq p(ab C^a C^b) - (p(a) + p(b)) \sqrt{\epsilon}, \end{aligned} \quad (104)$$

$$\begin{aligned} p(+_a +_b | ab) &\equiv \frac{p(+_a +_b ab)}{p(ab)} \\ &\geq p(C^a C^b) - \frac{(p(a) + p(b)) \sqrt{\epsilon}}{p(ab)}. \end{aligned} \quad (105)$$

(103) and (105) imply

$$p(+_a +_b | ab) - \Delta_{a,b}^+ < p(C^a C^b) \leq p(+_a +_b | ab) + \Delta_{a,b}^-, \quad (106)$$

with

$$\begin{aligned}\Delta_{a,b}^- &= \frac{(p(a) + p(b)) \sqrt{\epsilon}}{p(ab)}, \\ \Delta_{a,b}^+ &= \frac{(p(a) + p(b)) (5\sqrt{\epsilon} - 2\epsilon)}{p(ab)}.\end{aligned}\quad (107)$$

C Proof of the Clauser-Horne Inequality

In this appendix we will prove inequality (48). We consider arbitrary four events A , A' , B , and B' together with their complements. The sum over all 16 possibilities equals one:

$$\sum_{a,a',b,b'} p(a, a', b, b') = 1, \quad \text{where } a \in \{A, \bar{A}\} \text{ etc.} \quad (108)$$

We also have

$$\begin{aligned}p(AB) &= p(AA'BB') + p(AA'B\bar{B}') + p(A\bar{A}'BB') + p(A\bar{A}'B\bar{B}'), \\ p(AB') &= p(AA'BB') + p(A\bar{A}'BB') + p(AA'\bar{B}B') + p(A\bar{A}'\bar{B}B'), \\ p(A'B') &= p(AA'BB') + p(AA'\bar{B}B') + p(\bar{A}A'BB') + p(\bar{A}A'\bar{B}B'), \\ p(A'B) &= p(AA'BB') + p(AA'B\bar{B}') + p(\bar{A}A'BB') + p(\bar{A}A'B\bar{B}'), \\ p(A) &= p(AA'BB') + p(AA'B\bar{B}') + p(A\bar{A}'BB') + p(A\bar{A}'B\bar{B}') \\ &\quad + p(AA'\bar{B}B') + p(AA'\bar{B}\bar{B}') + p(A\bar{A}'\bar{B}B') + p(A\bar{A}'\bar{B}\bar{B}'), \\ p(B') &= p(AA'BB') + p(A\bar{A}'BB') + p(AA'\bar{B}B') + p(A\bar{A}'\bar{B}B') \\ &\quad + p(\bar{A}A'BB') + p(\bar{A}A'\bar{B}B') + p(\bar{A}\bar{A}'BB') + p(\bar{A}\bar{A}'\bar{B}B').\end{aligned}\quad (109)$$

Thus

$$\begin{aligned}p(AB) + p(AB') + p(A'B') - p(A'B) - p(A) - p(B') \\ = - [p(AA'B\bar{B}') + p(AA'\bar{B}\bar{B}') + p(A\bar{A}'\bar{B}B') + p(A\bar{A}'\bar{B}\bar{B}') \\ + p(\bar{A}A'BB') + p(\bar{A}A'B\bar{B}') + p(\bar{A}\bar{A}'BB') + p(\bar{A}\bar{A}'\bar{B}B')].\end{aligned}\quad (110)$$

Because each term appears only once on the R.H.S. of equation (110), equation (108) implies the Clauser-Horne inequality:

$$-1 \leq p(AB) + p(AB') + p(A'B') - p(A'B) - p(A) - p(B') \leq 0. \quad (111)$$

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